REFERENCES

1. ARTYSHEV S. G., On invariants of a linear differential equation. In *Mathematical Simulation of Problems in the Mechanics of a Continuous Medium*, pp. 3–6. Energoatomizdat, 1989.
2. ADOMIAN G., Convergent series solution of nonlinear equations. *J. Comp. Appl. Math.* **11**, 2, 225–230, 1984.
3. CHAO YU QIN, An explicit energy-conserving numerical method for equations of the form $d^2x/dt^2 = f(x)$. *J. Comp. Phys.* **79**, 2, 473–476, 1988.

Translated by D.L.

J. Appl. Maths Mechs Vol. 55, No. 6, pp. 939–943, 1991
Printed in Great Britain.
0021-8928/91 $15.00+.00

ON THE EXISTENCE OF STATIONARY SOLITARY WAVES IN A ROTATING FLUID†

V. N. GALKIN and YU. A. STEPANYANTS

Nizhnii Novgorod

(*Received* 24 *August* 1990)

A mathematical proof that there are no stationary solutions of the soliton type is given for a number of equations related to Ostrovskii's equation which, in particular, describes the surface and internal waves in a rotating fluid. A physical interpretation of this fact is presented. It is shown that, in the case of a different character of the high frequency dispersion which corresponds, for example, to capillary waves on a shallow rotating fluid, the conditions of the theorem are not satisfied as a result of which the prohibition on the existence of solitons is lifted. In this case, both single solitons as well as stationary formations consisting of solitons, that is, multisolitons, are constructed using numerical calculations.

1. FORMULATION OF THE PROBLEM

CONSIDER the class of non-linear wave equations of the form

$$\frac{\partial}{\partial x}\left(\frac{\partial\eta}{\partial t}+c\frac{\partial\eta}{\partial x}+\frac{\alpha}{p}\frac{\partial\eta^p}{\partial x}+\beta\frac{\partial^3\eta}{\partial x^3}\right)=\gamma\eta \tag{1.1}$$

Here, $\eta(x, t)$ is an unknown function, c, α, β, γ and p are constants and $p>1$. Equations belonging to this family are generated on the one hand by the generalised Korteweg–de Vries (KdV) equations and pass into them when $\gamma = 0$ and, on the other hand, their structure is close to the structure of the Kadomtsev–Petviashvili

†*Prikl. Mat. Mekh.* Vol. 55, No. 6, pp. 1051–1055, 1991.

(KP) equation. An equation belonging to the class (1.1) under consideration with $p = 2$ was derived for the first time by Ostrovskii [1] in order to describe internal waves in the ocean. Subsequently, analogous equations were obtained by many others for different types of waves (see the review [2] and the references contained therein).

The similarity between the generalized equation (1.1) and the classical KdV and KP equations, the distinctive feature of which is the occurrence of soliton solutions (by solitons, we mean solitary stationary waves without dwelling on questions of their stability, evolutionary character, etc.) also enables one to postulate the possibility of the existence of similar solutions within the framework of Eq. (1.1). It is found, however, that, in spite of the relative simplicity of this equation, it has not been possible up to now to give a complete description of its stationary solutions (the individual classes of such solutions, constructed using a digital computer, are given in [2]). An attempt was made in [3] to prove the fact that Eq. (1.1), when $p = 2$ and when the parameters β and γ have the same signs, does not possess soliton solutions. However, this proof, which is unnecessarily complex, contains a number of doubtful assertions, although it leads to the true result.

Below, we will put forward an extremely simple and rigorous proof of the absence of soliton solutions when $\beta\gamma>0$ not only strictly for the Ostrovskii equation but also in the case of its generalizations with an arbitrary value of p as well as for a number of other related equations (see [4, 5]).

We note that certain equations of this class have a direct relation to physical systems. In particular, when $p=3$, Eq. (1.1) describes the propagation of internal waves of even modes, which possess a cubic non-linearity, in the ocean.

2. THE "ANTISOLITON" THEOREM

By considering the stationary solutions of Eq. (1.1) which depend on a single variable $\xi = x - Vt$, we can rewrite it in the form of the system [3]

$$u'' = -\sigma p^{-1} u^p + au + bv, \quad v'' = u \tag{2.1}$$

$$u = \eta \left| \frac{\alpha}{\beta} \right|^{1/(p-1)}, \quad a = \frac{V-c}{\beta}, \quad b = \frac{\gamma}{\beta}, \quad \sigma = \begin{cases} 1, & \text{if } p \text{ is even} \\ \operatorname{sign}(\alpha/\beta), & \text{if } p \text{ is odd} \end{cases}$$

(the primes denote differentiation with respect to ξ). Next, we are interested in the soliton solutions of system (2.1) for which the function $u(\xi)\to 0$ when $|\xi|\to\infty$ together with its derivatives.

Let us prove that such solutions do not exist for a positive value of the parameter b. First of all, we note that system (2.1) possesses the integral

$$H^\circ = {}^1/_2\,[(u')^2 + b\,(v')^2] + \sigma p^{-1}(p+1)^{-1}u^{p+1} - {}^1/_2 a u^2 - buv \tag{2.2}$$

In a soliton solution far from its vertex which, to be specific, we shall put at the origin of coordinates, system (2.1) can be linearized by neglecting the term $\sim u^p$. A simple characteristic equation for solutions which are proportional to $l^{\lambda\xi}$ follows from the linear system which is obtained in this manner

$$\lambda^4 - a\lambda^2 - b = 0 \tag{2.3}$$

It can be shown that, for any sign of a and when $b>0$, it always has a pair of purely imaginary roots and a pair of real roots which are equal in modulus and opposite in sign. No further consideration will be given to the imaginary roots or to the localized solutions corresponding to them. In principle, the real roots could correspond to the asymptotic forms of the soliton solutions. Let us denote the real positive root by $\lambda_1 = [a/2 + \sqrt{(a^2/4 + b)}]^{1/2}$ and write the asymptotic forms of a possible soliton solution as

$$u(\xi) \sim \begin{cases} A\exp(\lambda_1\xi), & \text{when } \xi\to-\infty \\ B\exp(-\lambda_1\xi), & \text{when } \xi\to\infty \end{cases}$$

Let us next assume that the coefficient $A>0$ (<0). Then, $u(\xi)>0$ (<0) also when $\xi\to-\infty$. The fact that the second derivative of v vanishes at just a single point follows from the condition $v'(\pm\infty) = 0$ and Roll's theorem.

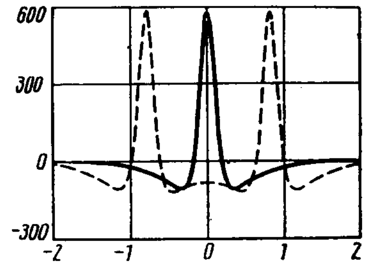

FIG. 1.

We select the smallest value of ξ_1 for which the equality $v''(\xi_1) = 0$ is satisfied. Then, in the interval $(-\infty, \xi_1)$, we will have $u(\xi) > 0$ (<0) and, also, $u(\xi_1) = 0$ at the point ξ_1 by virtue of the second equation of system (2.1). Returning to the integral (2.2), it may be concluded that $H = 0$ in a soliton solution. However, we then have that $u'(\xi_1) = v'(\xi_1) = 0$ at the point ξ_1. It is now possible to make use of Roll's theorem again and it follows from this that $v'' = 0$ at just the single point $\xi_2 \in (-\infty, \xi_1)$. However, this contradicts the assumption that ξ_1 is the smallest value for which $v'' = 0$. It follows from this that the assumption that a soliton solution exists with zero asymptotic forms at infinity is untrue.

3. SOLITON SOLUTIONS OF OSTROVSKII'S EQUATION WHEN $b<0$

The proof of the "antisoliton" theorem which has been presented in Sec. 2 is based essentially on the positiveness of the coefficient b and it no longer holds good when $b<0$. In the latter case, the search for stationary solutions of Ostrovskii's equation (with $p = 2$, $a = 75$ and $b = 1200$) using a digital computer and Petviashvili's method [6] leads to a soliton solution, the structure of which is represented by the solid line in Fig. 1. Here, on account of the presence of local extrema outside the vertex of the soliton, the formation of bound states consisting of two or more solitons, that is, multisoliton solutions, is possible. The stationary solution of Ostrovskii's equation for the same values of the parameters is shown in the form of a bisoliton by the dashed line.

We will now give a physical interpretation of the possibility of the existence of soliton solutions within the framework of Eq. (1.1). By linearizing (1.1) and seeking solutions of the resulting linear equation in the form $\eta \sim \exp(i\omega t - ikx)$, we find the corresponding dispersion relationship for waves of infinitely small amplitude

$$\omega = ck + \gamma/k - \beta k^3$$

An expression follows from this for the phase velocity ω/k. A plot of its dependence on k for different signs of β and $\gamma > 0$ is shown in Fig. 2. It is seen that, when $\beta > 0$, linear perturbations can exist over the whole range of

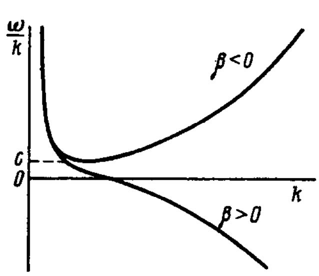

FIG. 2.

phase velocities from $-\infty$ to $+\infty$. As a consequence of this, an object moving with an arbitrary velocity in such a medium will inevitably be in resonance with some linear wave or other which leads to its excitation. Hence, a localized perturbation during the motion will experience radiative decay and this means that it will not be stationary.

However, these arguments cannot be considered as being completely rigorous or the generation of linear perturbations would now also depend on the structure of the moving source. In particular, soliton solutions with a complex internal structure are possible in certain cases when there is a resonance with linear perturbations [7, 8]. In the case of such solitons just a simultaneous generation and absorption of the excited waves in the internal domain would occur during the motion. Solutions of this kind usually form a finite or denumerable set and are unstable to small perturbations.

We also note that, when there is resonance, solutions are possible which represent solitons moving in a stationary manner against a background of periodic waves. The total energy of such perturbations is infinite and they are not considered here.

If, however, $\beta<0$, linear perturbations can only exist within a semibounded range of phase velocities when $w/k \geqslant c$. The possibility of solitons which are not in resonance with linear perturbations and are not subject to radiative decay then arises.

The parameter $\gamma>0$ [1–3] in the case of waves in a rotating fluid. Furthermore, in the case of surface gravitational and internal waves in the ocean, $\beta>0$ [1–3] and solitons are therefore impossible in the case of such waves. The parameter β can be negative in the case of capillary waves on the surface of a shallow rotating fluid or in the case of rapid magnetosonic waves in a rotating magnetized plasma [2]. The existence of not only one-dimensional but also two-dimensional multisolitons is possible in the case of such waves.

4. THE "ANTISOLITON" THEOREM FOR SHRIRA'S EQUATION

The somewhat more-general equation [4, 5]

$$v_{tt} - c^2 v_{xx} + \Omega^2 v - \frac{1}{3}\beta v_{ttxx} = [v_t v_x (\Omega + v_x)^{-1}]_t + \frac{1}{2}\Omega[(v_t)^2(\Omega + v_x)^{-2}]_x \tag{4.1}$$

is also considered together with Ostovskii's equation in the theory of linear waves in a rotating fluid.

In the case of stationary waves, this equation takes the form

$$[Qw''' - Pw' + (3/2 + w')(w')^2(1 + w')^{-2}]' = Rw \tag{4.2}$$

$$w = v/\Omega, \quad Q = \beta/3, \quad P = 1 - c^2/V^2, \quad R = \Omega^2/V^2$$

where, as previously, a prime denotes differentiation with respect to $\xi = x - Vt$.

This equation possesses a first integral:

$$H^0 = 1/2Q^{-1}\{Rw^2 + Q(w'')^2 - 2Qw'w''' + (w')^2[(1 + w')^{-2} + P - 1]\} \tag{4.3}$$

First of all, we note that, as follows directly from Eq. (4.2), it cannot have solutions of a shock-wave type (kinks). Actually, the left-hand side of this equation (and this means w also) vanishes when $|\xi|\to\infty$ for such solutions. Let us now prove that, when $RQ>0$, there also cannot be smooth soliton solutions of the pulse type with null asymptotic forms at infinity. In fact, as can be seen from (4.3), $H^0=0$ for such solutions. If it is assumed that a smooth soliton solution exists, then the equality $w'=0$ must be satisfied at the points of an extremum but we then get from (4.3) that $Rw^2+Q(w'')^2=0$ at the above-mentioned points. In other words, the solution must be trivial: $w(\xi)=0$.

The physical treatment of the absence of solitons in the case of Eq. (4.2) with $RQ>0$ is again a consequence of the fact that, within the framework of Eq. (4.1), linear perturbations can have any phase velocity in a range from $-\infty$ to $+\infty$.

5. CONCLUSION

Apart from the equations which have been considered here, there are other related equations which possess similar properties for which the presence or absence of soliton solutions is determined by the ratio of the coefficients characterizing the low- and high-frequency dispersion. A rigorous mathematical proof of the "antisoliton" theorem cannot be given for all of these equations. However, the physical considerations presented in Sec. 3 are usually also applicable to these equations. As an example, let us consider the equation which describes the non-linear internal waves in a deep rotating ocean [9]

$$\frac{\partial}{\partial x}\left(\frac{\partial\eta}{\partial t}+c\frac{\partial\eta}{\partial x}+\alpha\eta\frac{\partial\eta}{\partial x}+\delta\frac{\partial^2}{\partial x^2}\mathbf{H}\eta\right)=\gamma\eta$$

$$\mathbf{H}\eta=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\eta(\zeta,t)}{x-\zeta}\,d\zeta \tag{5.1}$$

(the integral is to be understood in the sense of a principal values).

When searching for stationary solutions, it is convenient to represent this equation in the form of a system analogous to (2.1):

$$(\mathbf{H}u)'=-\frac{1}{2}u^2+au+bv,\quad v''=u$$

$$u=\alpha\eta/\delta,\quad a=(V-c)/\delta,\quad b=\gamma/\delta \tag{5.2}$$

(the prime denotes differentiation with respect to $\xi=x-Vt$). In the case of (5.2), the first integral is

$$H^\circ=\int_{-\infty}^{\xi}u'(\mathbf{H}u)'\,d\xi+\frac{1}{2}b(v')^2+\frac{1}{6}u^3-\frac{1}{2}au^2-buv \tag{5.3}$$

However, it is not possible to use this integral to prove the absence of soliton solutions in the case of system (5.2) in the spirit of what was done in Sec. 2 on account of the fact that the first integral term on the right-hand side of (5.3) is undetermined with respect to its sign. Physical considerations based on an analysis of the form of the dispersion curve $\omega=ck+\gamma/k-\delta k^2$ suggest that, when $b<\gamma/\delta>0$ (a situation characteristic of internal oceanic waves) there must also not be any soliton solutions.

REFERENCES

1. OSTROVSKII L. A., Non-linear internal waves in a rotating ocean. *Okeanologiya* **18**, 2, 181–191, 1978.
2. OSTROVSKII L. A. and STEPANYANTS Yu. A., Non-linear surface and internal waves in a rotating fluid. In *Non-Linear Waves*. Nauka, Moscow, 1991.
3. LEONOV A. I., The effect of Earth rotation on the propagation of weak nonlinear surface and internal long oceanic waves. *Annals New York Acad. Sci.* **373,** 150–159, 1981.
4. SHRIRA V. I., Propagation of long non-linear waves in a layer of a rotating fluid. *Izv. Akad. Nauk SSSR, Fiz. Atmosfery i Okeana* **17,** 1, 76–81, 1981.
5. SHRIRA V. I., On long essentially non-linear waves in a rotating ocean. *Izv. Akad. Nauk SSSR, Fiz. Atmosfery i Okeana* **22,** 4, 395–405, 1986.
6. PETVIASHVILI V. I. and POKHOTELOV O. A., *Solitary Waves in a Plasma and in the Atmosphere*. Energoatomizdat, Moscow, 1989.
7. KOZLOV V. A., LITVAK A. G. and SUVOROV E. V., Solitons of envelopes of strong relativistic electromagnetic waves. *Zh. Eksp. Teor. Fiz.* **76,** 1, 148–157, 1979.
8. BOGOMOLOV Ya. L., KOL'CHUGINA I. A., LITVAK A. G. and SERGEYEV A. M., Near-sonic Langmuir solitons. *Phys. Lett.* **94A,** 9, 447–450, 1982.
9. GRIMSHAW R., Evolution equations for weakly nonlinear long internal waves in a rotating fluid. *Stud. Appl. Math.* **73,** 1–33, 1985.

Translated by E.L.S.